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LETTER TO THE EDITOR

New insights in particle dynamics from group cohomology

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Abstract
The dynamics of a particle moving in background electromagnetic and gravitational fields is revisited from a Lie group cohomological perspective. Physical constants characterizing the particle appear as central extension parameters of a group which is obtained from a centrally extended kinematical group (Poincaré or Galilei) by making some subgroup *local*. The corresponding dynamics is generated by a vector field inside the kernel of a pre-symplectic form which is derived from the canonical left-invariant 1-form on the extended group. A *non-relativistic* limit is derived from the *geodesic* motion via an Inönü–Wigner contraction. A deeper analysis of the cohomological structure reveals the possibility of a new force associated with a non-trivial mixing of gravity and electromagnetism leading to, in principle, testable predictions.

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1. General setting

The aim of this work is to clarify the underlying algebraic structure behind the dynamics of a particle moving inside a background field. We shall see how the constants characterizing the properties of the particle and its couplings can be understood in terms of the parameters associated with the central extensions of certain groups, thus bringing into scene group-cohomological concepts.

In order to motivate the role of central extensions, let us first recall some basic and well-known facts with the help of an example representing, perhaps, the simplest physical system one can imagine: the free particle with Galilei symmetry. In order to define the system

we make use of the *Poincaré–Cartan* form defined on what we shall call *evolution* space constructed from phase space by adding time, (x, p, t),

$$\Theta_{PC} = p \, dx - \frac{p^2}{2m} \, dt. \tag{1}$$

A realization of Galilei group, parametrized by (b, a, V) (we omit rotations) and the corresponding infinitesimal transformations (generators), on this evolution space is the following:

$$
\begin{aligned}
x' &= x + a + Vt & X_a &= \frac{\partial}{\partial x} \\
t' &= t + b & X_b &= \frac{\partial}{\partial t} \\
p' &= p + mV & X_V &= m\frac{\partial}{\partial p} + t\frac{\partial}{\partial x}.
\end{aligned}
\tag{2}
$$

When checking the invariance of the Poincaré–Cartan form under the Galilei group we realize that its variation under the action of *boosts* is a total differential, rather than zero. In infinitesimal terms, the Lie derivative

$$L_{X_V}\Theta_{PC} = d(mx) \neq 0 \tag{3}$$

thus leading to the idea of semi-invariance. Of course, this is not a problem at the classical level, since the equations of motion are not sensitive to such a total derivative variation. Although it is not necessary in a strict manner, let us raise to the level of a postulate the claim for strict invariance and let us see what consequences we can derive from this assumption.

In order to achieve such strict invariance, let us extend the evolution space with a new variable $\eta = e^{i\phi}$ which transforms under the Galilei group in such a way that the variation of the total differential of ϕ inside a modified 1-form, $\Theta \equiv \Theta_{PC} + d\phi$, cancels out the term $d(mx)$. That is

$$d\phi' = d\phi - d(mx). \tag{4}$$

The corresponding finite action of the Galilei group on this new variable is then (a new group parameter φ must be included for the following expression to be a proper action)

$$\eta' = \eta \, e^{-i[\frac{1}{2}mV^2t + mVx + \varphi]} \tag{5}$$

which together with the action on the rest of the variables in the extended evolution space (2), allows us to compute the infinitesimal generators as well as their commutation relations:

$$[\tilde{X}_b, \tilde{X}_a] = 0 \qquad [\tilde{X}_b, \tilde{X}_V] = \tilde{X}_a \qquad [\tilde{X}_a, \tilde{X}_V] = m\tilde{X}_\varphi. \tag{6}$$

We note the presence of a central term in the last commutator, and, therefore, the claim for strict invariance has led us to the centrally extended Galilei group.

The crucial fact about this phenomenon leading to the central extension is that it is not related to a particular realization of the group, but it is a consequence of its intrinsic algebraic structure, in fact related to group-cohomologial features. This suggests considering the group itself (a centrally extended group, in fact) as the starting point in the definition of the dynamics of a physical system.

This is exactly the aim of the so-called group approach to quantization (GAQ) (see [1] and references therein), which tries to derive a dynamics directly from the strict symmetry of the corresponding physical system where the group-cohomology plays a central role. Even though the main stress of the approach leans on its quantum aspects, it also has non-trivial implications at the classical level, which are the aspects we are going to emphasize here.

The extended evolution space and the Poincaré–Cartan form are generalized by objects that can be completely recovered from the centrally extended symmetry group. The latter has the structure of a principal fibre bundle $\tilde{G}$ whose base is the non-extended group G and where the fibre is the $U(1)$ group of phase invariance of quantum mechanics. The relevant cohomology in the construction is that of G. The object generalizing the Poincaré–Cartan form is the component of the left-invariant canonical 1-form on the group which is dual to the vertical (or central) vector field: $\Theta = \theta^{L^{(\zeta)}}$. By its own construction, this Θ generalizing $\Theta_{PC} + \mathrm{d}\phi$ (which we will call quantization 1-form) is invariant under the left action of the extended group (meanwhile Θ_{PC} is semi-invariant under $\tilde{G}/U(1)$). Regarding classical dynamics, the form $\mathrm{d}\Theta$ can be seen as a pre-symplectic form, in such a way that the solution space (i.e. the phase space) is obtained from the group by getting rid of those variables inside the kernel of $\mathrm{d}\Theta$. In this way, the trajectories of the vector fields inside this kernel can be seen as generalized equations of motion [2]. In principle, there is a certain ambiguity in the choice of the Hamiltonian vector field, but this problem will not arise in the system we shall consider here.

Let us illustrate this technique revisiting the free Galilean particle under this perspective. For instance, from the realization of the group on the evolution phase (2) and (5), we can derive the following group law for the centrally extended Galilei group:

$$
\begin{aligned}
b'' &= b' + b & a'' &= a' + a + V'b \\
V'' &= V' + V & \zeta'' &= \zeta'\zeta\, \mathrm{e}^{-\mathrm{i}\frac{m}{\hbar}[V'a + \frac{1}{2}bV'^2]}.
\end{aligned}
\tag{7}
$$

We can use it to compute the right- and left-invariant vector fields (from now on, we shall write x for a, t for b, v for V, φ for ϕ and therefore ζ for η, whenever the discussion takes place in the GAQ setting, and set $\hbar = 1$)

$$
\begin{aligned}
\tilde{X}_t^L &= \frac{\partial}{\partial t} + v\frac{\partial}{\partial x} - \frac{1}{2}mv^2\frac{\partial}{\partial\varphi} & \tilde{X}_t^R &= \frac{\partial}{\partial t} \\
\tilde{X}_x^L &= \frac{\partial}{\partial x} - mv\frac{\partial}{\partial\varphi} & \tilde{X}_x^R &= \frac{\partial}{\partial x} \\
\tilde{X}_v^L &= \frac{\partial}{\partial v} & \tilde{X}_v^R &= \frac{\partial}{\partial v} + t\frac{\partial}{\partial x} - mx\frac{\partial}{\partial\varphi} \\
\tilde{X}_\varphi^L &= \frac{\partial}{\partial\varphi} & \tilde{X}_\varphi^R &= \frac{\partial}{\partial\varphi}
\end{aligned}
\tag{8}
$$

and the quantization 1-form

$$
\Theta \equiv \theta^{L^\varphi} = mv\,\mathrm{d}x - \tfrac{1}{2}mv^2\,\mathrm{d}t + \mathrm{d}\varphi
\tag{9}
$$

where the first two terms on the rhs correspond to the original Poincaré–Cartan form.

2. Interactions

2.1. Electromagnetism

Up to now we have sketched two alternative ways of considering the role of the symmetry group when dealing with the dynamics of a physical system. The first one makes use of a particular realization of the group on an extended phase space, while the second one (that of GAQ) emphasizes the singular role of the group, considering it as the departing point in the analysis. At this point we switch to interactions, starting with the electromagnetic force, and study the situation from the perspective of both technical strategies.

A natural question arising when considering the centrally extended Galilei group refers to the consequences of making the $U(1)$ part of the symmetry local[4]. Dwelling on the setting of the first approach [4], the Lie algebra with a local $U(1)$ is composed of the former Galilei generators realized on the extended evolution space, together with the tensor product of local functions and the central term: $f(x, t) \otimes X_\phi$. But when we check the invariance of the modified Poincaré–Cartan form $(\Theta_{PC} + d\phi)$ under the new generators, semi-invariance reappears:

$$L_{f \otimes X_\phi} \Theta = df. \tag{10}$$

We follow here the same strategy as before, i.e. we look for new variables extending the evolution space whose variation under the symmetry group compensates the variation df in a newly modified 1-form Θ. Fortunately, in this case there are natural guesses and we are able to find new variables A_0, A_x, transforming in the desired way $(A' = A - df)$.

The realization of the fields in the newly extended phase space $(x, p, t, \phi, A_0, A_x)$ is

$$
\begin{aligned}
\tilde{X}_b &= \frac{\partial}{\partial t} \\
\tilde{X}_a &= \frac{\partial}{\partial x} \\
\tilde{X}_v &= t\frac{\partial}{\partial x} + m\frac{\partial}{\partial p} - mx\frac{\partial}{\partial \phi} + A_x\frac{\partial}{\partial A_0} \\
f\widetilde{\otimes X}_\phi &= -f\frac{\partial}{\partial \phi} - \frac{\partial f}{\partial x}\frac{\partial}{\partial A_x} + \frac{\partial f}{\partial t}\frac{\partial}{\partial A_0}.
\end{aligned}
\tag{11}
$$

The new strictly invariant 1-form is

$$\Theta = p\, dx - \frac{p^2}{2m}\, dt - A_x\, dx + A_0\, dt + d\phi \tag{12}$$

from which the Lorentz force felt by the particle can be straightforwardly derived.

As an alternative approach, we apply the techniques of GAQ to this problem, which results in a more algorithmic and general treatment. In fact, if we consider an arbitrary group $\tilde{G}$ whose infinitesimal generators are $\{X_\alpha\}$ $(\alpha = 1, \dots, n)$ and an invariant subgroup $\{X_i\}$ $(i = 1, \dots, m < n)$, we can make the latter *local*, obtaining an algebra spanned by $\{f \otimes X_i, X_\alpha\}$ and whose new commutators are

$$[X_\alpha, f \otimes X_i] = f \otimes [X_\alpha, X_i] + L_{X_\alpha} f \otimes X_i = f \otimes C_{\alpha i}^j X_j + L_{X_\alpha} f \otimes X_i. \tag{13}$$

One then applies GAQ tools to obtain the quantization 1-form Θ.

In the case of the particle inside the electromagnetic field, we are dealing with the Galilei group extended by $U(1)(\vec{x}, t)$, that is, $\varphi = \varphi(\vec{x}, t)$. This algebra is infinite-dimensional but, for analytical functions f, we can resort to the following economical short cut. We start by adding to Galilei algebra only those generators $f \otimes \tilde{X}_\varphi$ for which f are linear functions, i.e. $t \otimes \tilde{X}_\varphi$ and $x^i \otimes \tilde{X}_\varphi$, to be referred to as $\tilde{X}_{A^0}$ and $\tilde{X}_{A^i}$, respectively. Let us call $\tilde{G}_E$ the group associated with this finite-dimensional algebra. The commutation relations of $\tilde{G}_E$ are (omitting zero commutators as well as rotations, which operate in the standard way)

$$
\begin{array}{ll}
[\tilde{X}_{v^i}, \tilde{X}_t] = \tilde{X}_{x^i} & [\tilde{X}_{x^i}, \tilde{X}_{v^j}] = -m\delta_{ij}\tilde{X}_\varphi \\
[\tilde{X}_t, \tilde{X}_{A^0}] = -q\tilde{X}_\varphi & [\tilde{X}_{x^i}, \tilde{X}_{A^j}] = q\delta_{ij}\tilde{X}_\varphi \\
[\tilde{X}_{v^i}, \tilde{X}_{A^i}] = \delta_{ij}\tilde{X}_{A^0}
\end{array}
\tag{14}
$$

---

[4] Intuition is led by the fact that a local $U(1)$ group is intrinsically related to electromagnetism by means of a minimal coupling principle [3].

where we have performed a *new central extension*, which is allowed by the Jacobi identity and parametrized by q. This parameter will eventually be identified (see below) with the electric charge of the particle.

Fortunately, $\tilde{G}_E$ encodes all the dynamical information in the local group. The only effect of including all functions f is that of making local the group parameters A^0 and A^i (corresponding to $\tilde{X}_{A^0}$ and $\tilde{X}_{A^i}$, respectively), something we shall recover here exactly by imposing an appropriate constraint. After the exponentiation of the group, we compute the quantization 1-form which turns out to be

$$\Theta = m\vec{v} \cdot d\vec{x} + q\vec{A} \cdot d\vec{x} - \left(\tfrac{1}{2}m\vec{v}^2 + q A_0\right) dt + d\varphi \tag{15}$$

which is again the Poincaré–Cartan of a particle inside an electromagnetic field plus the central term $d\varphi$. In order to derive the equations of motion of the particle we have to impose the above-mentioned constraint, making the functions A^μ (A^i and A^0) to depend on the position of the particle: $A = A(x_{\text{particle}})$. The vector field X in the kernel of Θ, that is, satisfying $i_X d\Theta = 0$ is

$$X = \frac{\partial}{\partial t} + \vec{v} \cdot \frac{\partial}{\partial \vec{x}} - \frac{q}{m}\left[\left(\frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}\right)v^j + \frac{\partial A_0}{\partial x^i} + \frac{\partial A_i}{\partial t}\right]\frac{\partial}{\partial v_i} \tag{16}$$

and its trajectories are governed by the following equations:

$$\frac{d\vec{x}}{dt} = \vec{v} \qquad m\frac{d\vec{v}}{dt} = q\left[\vec{v} \wedge (\vec{\nabla} \wedge \vec{A}) - \vec{\nabla}A_0 - \frac{\partial \vec{A}}{\partial t}\right] \tag{17}$$

which result in the standard expression for the Lorentz force, when we define $\vec{\nabla} \wedge \vec{A} \equiv \vec{B}$ and $-\vec{\nabla}A_0 - \frac{\partial \vec{A}}{\partial t} \equiv \vec{E}$. We have studied this electromagnetic example in the Galilean scheme for pedagogical reasons, but we must stress that everything can be reproduced in the relativistic case, starting from the Poincaré group and constructing the corresponding $\tilde{\mathcal{P}}_E$, finally leading to the same final expression.

2.2. Electromagnetism and gravity mixing

We now address a more involved system using for its analysis the most algorithmic of the two techniques we have presented up to now: GAQ.

We start directly with the centrally extended Poincaré group and try to make the spacetime translation subgroup local, instead of the central $U(1)$ one. These local translations can be seen as local diffeomorphisms, thus suggesting the emergence of gravity notions [5]. The Lie algebra can be written as

$$[\tilde{X}_t, \tilde{X}_{v^i}] = \tilde{X}_x \qquad [\tilde{X}_{x^i}, \tilde{X}_{v^j}] = -\delta_{ij}(\tilde{X}_t + m\tilde{X}_\varphi). \tag{18}$$

An interesting phenomenon is observed when we make the spacetime translations local. In fact, when computing the commutators following the above-mentioned general rule, we find

$$[\tilde{X}_{v^i}, f \otimes \tilde{X}_{x^j}] = (X_{v^i}f) \otimes \tilde{X}_{x^j} + \delta_{ij}(f \otimes \tilde{X}_t + f \otimes \tilde{X}_\varphi). \tag{19}$$

This means that making the translation generators local in the extended Poincaré group implies the appearance of a local $U(1)$ symmetry. This fact is linked to the loss of invariant character of the translation subgroup in the extended Poincaré group. As we saw in the previous subsection, making the central term local leads to a coupling between the particle and an electromagnetic force. Therefore, we find that introducing the gravitational field offers the quite interesting possibility of an automatic coupling of the particle and an electromagnetic field and suggests the possibility of a mixing between both interactions.

In order to derive the corresponding dynamics we undertake exactly the same path as we follow in the pure electromagnetism case. (Symmetrized) generators of local spacetime translations associated with the linear functions, $x_\mu \otimes P_\nu + x_\nu \otimes P_\mu$, will be called $X_{h^{\mu\nu}}$, and close the finite-dimensional algebra $\tilde{\mathcal{P}}_{EG}$ ($\supset \tilde{\mathcal{P}}_E$) analogous to $\tilde{\mathcal{G}}_E$. The analogue of the vector A^μ is now a symmetric tensor $h^{\mu\nu}$, eventually interpreted as a metric. However, the central-charge structure of this finite-dimensional electro-gravitational subgroup, $\tilde{P}_{EG}$, is richer than that of $\tilde{P}_E$ or $\tilde{G}_E$, entailing a more involved GAQ analysis which must be made, for the time being at least, order by order. For the sake of clarity, we shall dwell here on a simplified case corresponding to a *non-relativistic* limit, which still preserves the main features of the discussion. This limit is achieved by means of an Inönü–Wigner contraction of $\tilde{\mathcal{P}}_{EG}$ with respect to the subalgebra spanned by $\tilde{X}_t$, $\tilde{X}_{A^i}$ and the rotations. The contracted algebra shows (omitting rotations)

$$
\begin{aligned}
[\tilde{X}_t, \tilde{X}_{v^i}] &= P_i & [\tilde{X}_t, \tilde{X}_{h^{00}}] &= -2m\tilde{X}_\varphi \\
[\tilde{X}_t, \tilde{X}_{h^{0i}}] &= \tilde{X}_{x^i} & [\tilde{X}_t, \tilde{X}_{A^0}] &= q\tilde{X}_\varphi \\
[\tilde{X}_{x^i}, \tilde{X}_{v^j}] &= -(m + \kappa q)\delta_{ij}\tilde{X}_\varphi & [\tilde{X}_{x^i}, \tilde{X}_{A^j}] &= -q\delta_{ij}\tilde{X}_\varphi \\
[\tilde{X}_{x^i}, \tilde{X}_{h^{0j}}] &= m\delta_{ij}\tilde{X}_\varphi & [\tilde{X}_{v^i}, \tilde{X}_{h^{0j}}] &= -\delta_{ij}\tilde{X}_{h^{00}} + \kappa\delta_{ij}\tilde{X}_{A^0} \\
[\tilde{X}_{v^i}, \tilde{X}_{A^j}] &= -\delta_{ij}\tilde{X}_{A^0} & [\tilde{X}_{h^{0i}}, \tilde{X}_{A^j}] &= -\delta_{ij}\tilde{X}_{A^0}.
\end{aligned}
\tag{20}
$$

The most significant characteristic of this algebra is the appearance of a new constant κ associated with the already commented on mixing of electromagnetic and gravity forces (this fact is apparent in the non-contracted algebra where commutators of the type $[\tilde{X}_{h^{\mu\nu}}, \tilde{X}_{h^{\alpha\beta}}] \sim \tilde{X}_{A^\rho}$ are present). The appearance of this κ in the commutator between boosts and translations, in fact, modifies the inertial mass by a term κq, something most relevant from the physical point of view.

The next step requires the exponentiation of the algebra and the subsequent construction of the quantization 1-form from which the equations of motion can be derived. The exponentiation process is still complicated and we employ a consistent order-by-order procedure which can be found in [6]. The following equations of motion correspond to the first non-trivial terms approximating the complete equations:

$$
\frac{d\vec{x}}{dt} = \vec{v}
$$

$$
\begin{aligned}
(m + \kappa q)\frac{d\vec{v}}{dt} = {} & q\left[\vec{v} \wedge (\vec{\nabla} \wedge \vec{A}) - \vec{\nabla}A^0 - \frac{\partial\vec{A}}{\partial t}\right] \\
& - m\left[\vec{v} \wedge (\vec{\nabla} \wedge \vec{h}) - \vec{\nabla}h^{00} - \frac{\partial\vec{h}}{\partial t}\right] + \frac{m}{4}\nabla(\vec{h} \cdot \vec{h}) \\
& - \frac{\kappa q}{2}\left[\vec{v} \wedge (\vec{\nabla} \wedge \vec{h}) - \frac{1}{4}\nabla(\vec{h} \cdot \vec{h}) - \frac{\partial\vec{h}}{\partial t}\right].
\end{aligned}
\tag{21}
$$

In the kinematical part (lhs), we explicitly note what was already foreseen at the Lie-algebra level, i.e. the kinematical mass is corrected by a term proportional to κ and the charge of the particle.

On the dynamical side, the first line is again the expression of the Lorentz force, while the expression in the second line, known as *gravito-electromagnetism* [7], corresponds to the *geodesical motion* in its first non-trivial perturbative expression (linearized gravity), which is that obtained when working in the group law up to the third order in group variables as we have done. The last line is proportional to the mixing parameter κ and shows the appearance

of a new force of *electromagnetic* behaviour but of *gravitational* origin; a consequence of this new possibility opened by the analysis of the underlying symmetry cohomology.

3. Conclusions

We have seen how physical constants characterizing the particle and its couplings arise from the parameters associated with the cohomology of the symmetry underlying the physical system. This was already well known for the mass m and it is explicitly shown here for the electric charge q.

We have recovered in an algebraic setting two standard interactions, electromagnetism and gravity, by making local some invariant subgroups in the *kinematical* symmetry of the particle. This in fact constitutes a revision of the gauge principle in this framework. When exploring the full possibilities that the Lie algebra permits, a new force parametrized by a constant κ has also been found associated with a mixing of the standard previous interactions. Nature could choose κ to be zero, but the presence of this new term is a possibility that the Lie algebra definitely offers. A crucial observation of this mixing process is the relevance of making the symmetry local (translations) *after* a central extension of the group has already been performed. This endows the new interaction with a *quantum flavour* since such a central extension is intrinsically tied to the quantum symmetry of the corresponding system [8].

A value of κ different from zero has, in principle, two direct testable consequences. Firstly, it entails a $2\kappa q$ mass difference between charged particles and anti-particles. An algebraic treatment such as the present one is not primarily related to additional physical phenomena such as radiative corrections, but it in turn offers a conceptual algebraic framework in the case that these effects actually occur. In that case, the current experimental clearance in the values of mass differences in pairs such as electron–positron represents an upper bound for the constant κ, implying a very small value (around $10^{-8}\, m_e$). Nevertheless, this tiny value would have strong and fundamental implications, especially the violation of CPT symmetry.

Secondly, since κ modifies the inertial mass (lhs in (21)) but leaves untouched the gravitational mass (on the rhs) it represents an explicit violation of the *weak equivalence principle*, a result with far-reaching conceptual consequences.

Acknowledgment

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